Effects of anisotropy on convection in porous media subject to nonuniform thermal average gradient

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Abstract - The objective of the present study is to investigate analytically the effect of anisotropy on the onset of natural convection heat transfer in a fluid saturated porous horizontal cavity subjected to nonuniform thermal gradients, taking into account the hydrodynamic anisotropy of the porous matrix and the rigid/rigid and stress-free/stress-free horizontal boundaries. We have obtained the exact solutions for the flow and heat transfer variables, valid for the onset of convection related to vanishingly small wave number and depending on the Darcy-Rayleigh number, the Darcy number, the anisotropic permeability ratio and the inclination of the principal axis. Furthermore the critical Rayleigh number for the onset of convection is computed and the cases when the Darcy number approaches ∞ or 0 are discussed. The results of this paper are the generalization of the results obtained by Degan and Vasseur.

Key words: Anisotropy, Boundary conditions, Convection, Porous matrix.

Nomenclature

1. a, b, c constants
2. C dimensionless temperature gradient in x direction
3. Da Darcy number , K/L^2
4. $\vec{g}$ gravitational acceleration
5. H' depth of cavity
6. K thermal conductivity
7. $\overline{K}$ flow permeability tensor
8. K_1, K_2 flow permeability along the principal axes
9. K^* anisotropic permeability ratio, K_1/K_2
10. L' width of cavity
11. Nu Nusselt number
12. q' uniform heat flux
13. Ra Darcy-Rayleigh number, $g\beta K_1 L'^2 q'/(k\nu\alpha)$
14. Ra^* Rayleigh number for a fluid, Ra/Da
15. Ra_c critical Rayleigh number for a porous medium
16. $Ra_c{}^*$ critical Rayleigh number for a fluid
17. t dimensionless time
18. $\tilde{T}$ dimensionless temperature
19. T dimensionless quasi-state temperature, $\tilde{T} - \xi t$
20. $\Delta T'$ temperature scale,$q'L'/k$
21. ΔT wall to wall dimensionless temperature difference at $x = 0$
22. $\vec{V}$ seepage velocity
23. u, v dimensionless velocity components in x, y directions
24. x dimensionless horizontal coordinate
25. y dimensionless vertical coordinate Greek symbols
26. α thermal diffusivity
27. β thermal expansion coefficient of the fluid
28. γ inclination of the principal axis
29. μ dynamic viscosity of the fluid
30. ν kinematic viscosity of the fluid
31. ϕ y - dependent temperature term
32. ψ dimensionless stream function , ψ'/α
33. ρ density of the fluid
34. $(\rho C)_f$ heat capacity of the fluid
35. $(\rho C)_p$ heat capacity of saturated porous medium
36. τ dimensionless Darcy parameter, $Da^{-\frac{1}{2}}$
37. σ heat capacity ratio, $(\rho C)_p/(\rho C)_f$
38. ξ dimensionless uniform heat sink Superscript
39. $'$ dimensional quantities Subscript
40. o refers to origin

1. INTRODUCTION

Natural convection heat transfer in a fluid saturated porous horizontal cavity subjected to nonuniform thermal gradients with the hydrodynamic anisotropy of the porous matrix and the rigid/rigid and stress-free/stress-free horizontal boundaries remains one of the most important problems for modern theoretical physics and applied mathematics. This problem is motivated by engineering applications as: convection in the Earth's crust, flows in soils, aquifers, storage of agriculture products and so on.

Since complicated convective motions appear in the layers near the surface, many scientific papers aim at investigating the conditions for stability or instability. Most of the work on onset of convection in a porous medium is based on linear theory.The critical Rayleigh number derived by such a theory gives a necessary condition for stability(or equivalently, a sufficient condition for instability). Degan and Vasseur [6] have studied the influence of anisotropy on convection in porous media with nonuniform thermal gradient by assuming the flow parallel in the x-direction. The stability of convection in a horizontal porous layer subjected to an inclined gradient of finite amplitude was investigated by Weber [12] and Nield [9] . The results showed by the critical Rayleigh number are always higher than $4\pi^2$. The stability of horizontal porous and viscous layer, when the thermal gradient is not uniform, has been considered by Nield [9]. Walker and Homsy [11] have used the Brinkman model to determine the critical Rayleigh number against Darcy number for the case of conducting no-slip boundaries. Vasseur and Robillard [10] have used the Brinkman model to investigate the effects of nonlinear temperature distribution on stability and natural convection in a horizontal porous layer heated from below. All previous studies have usually been concerned with homogeneous porous structures. But the inclusion of more physical realism in the matrix properties of the medium is important for the accurate modeling of the anisotropic media. Anisotropy, which is generally a consequence of a preferential orientation or asymmetric geometry of the grain of fibers, is in fact encountered in numerous systems in industry and nature. The critical Rayleigh number for the onset of convection was first considered by Castinel and Combarnous [4] who conducted an experimental and theoretical investigation for a layer with impermeable boundaries.Mckibbin [8] conducted an extensive study on the effects of anisotropy on the convective stability of a porous layer.

Also, it was demonstrated that the critical Rayleigh number was always reduced when compared with a perpendicular or parallel orientation of boundaries.From many papers it was found that anisotropic medium is most stable while either the isotropic situation or the horizontally isotropic situation is the most unstable one depending on the horizontal Rayleigh number and anisotropy parameters.

In this paper we aim at determining the critical Rayleigh numbers for the onset of convection on the basis of the generalized Brinkman-extended Darcy model which allows the no-slip boundary condition on a solid wall,to be satisfied.

The rest of this paper is organized as follows: in the next section we present the details of the model we will analyze. In a section 3 we look for the solutions to the governing equations, depending on the anisotropic parameters in permeability of the porous matrix and we compute the critical Rayleigh number. It is demonstrated that anisotropic parameters have considerable influence on the onset of convection. The last section contains the discussion and the conclusion .

2. Governing equations

The physical model considered here consists of a two dimensional horizontal rectangular enclosure of elongated shape filled with a porous medium composed of sparse distribution of particles completely surrounded by Boussinesq fluid and bounded by two rigid vertical side walls and two long horizontal boundaries at $y' = 0, y' = L$ that may be both rigid or both stress-free. The anisotropy in flow permeability of the porous medium is characterized by the permeability ratio $K^* = K_1/K_2$ (where K_1 and K_2 are the permeabilities along the two principal axes of the porous matrix respectively) and the orientation angle γ, defined as the angle between the horizontal direction and the principal axis with the permeability K_2. The layer is heated from the bottom by a constant heat flux q' and the other surfaces are insulated. We assume that the viscous,incompressible and saturating fluid and the porous medium are everywhere in local thermodynamic equilibrium.

Under the above approximations,the equations describing the laminar and two-dimensional convective flow in an anisotropic porous medium can be written in the form as in $[6]$ and $[1]$:

$$\nabla.\overrightarrow{V'} = 0 \quad (1)$$

$$\frac{\overline{K}}{\mu}\left(-\nabla p' + \rho\vec{g} + \mu_{eff}\nabla^2\vec{V}'\right) = \vec{V}' \quad (2)$$

$$(\rho c)_p \frac{\partial T'}{\partial t'} + (\rho c)_f \mathrm{x}\nabla.\left(T'\vec{V}'\right) = k\nabla^2 T' \quad (3)$$

$$\rho_0[1-\beta(T'-T'_0)]=\rho \quad (4)$$

where $\overrightarrow{V'}$ is the superficial flow velocity, T' the temperature, $\vec{g}$ the gravitational acceleration, t' the time, $(\rho c)_p$ and $(\rho c)_f$ the heat capacities of the saturated porous medium and the fluid respectively , μ the dynamic viscosity,

μ_{eff} apparent dynamic viscosity for Brinkman's model, k the thermal conductivity, ρ the density,

$$\overline{K}=\begin{pmatrix} K_2cos^2\gamma+K_1sin^2\gamma & (K_2-K_1)sin\gamma cos\gamma \\ (K_2-K_1)sin\gamma cos\gamma & K_1cos^2\gamma+K_2sin^2\gamma \end{pmatrix}$$

the symmetrical second-order permeability tensor and $\rho=\rho_0[1-\beta(T'-T'_0)]$ the Boussinesq approximation.

We take $L', \frac{\alpha}{L'}, \Delta T'=q'L'/k, \psi'/\alpha$ and $\sigma L'^2/\alpha$

$$\left(where\ \alpha=k/(\rho c)_f,\ \ \sigma=(\rho c)_p/(\rho c)_f\right)$$

as respective dimensional scales for length, velocity, temperature, stream function and time.

Taking into account the components of the velocity

$u=\frac{\partial\psi}{\partial y}, v=-\frac{\partial\psi}{\partial x}$ and applying the operator rot to the equation (2) we can write the governing equations in nondimensional form as

$$a\frac{\partial^2\psi}{\partial x^2}+c\frac{\partial^2\psi}{\partial x\partial y}+b\frac{\partial^2\psi}{\partial y^2}=\lambda Da\nabla^4\psi-Ra\frac{\partial\tilde{T}}{\partial x} \quad (5)$$

$$\frac{\partial^2\tilde{T}}{\partial x^2}+\frac{\partial^2\tilde{T}}{\partial y^2}=\frac{\partial\tilde{T}}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\tilde{T}}{\partial y}\frac{\partial\psi}{\partial x}+\frac{\partial\tilde{T}}{\partial t} \quad (6)$$

Where

$$\begin{cases} a=cos^2\gamma+K^*sin^2\gamma \\ b=sin^2\gamma+K^*cos^2\gamma \\ c=(1-K^*)sin2\gamma \end{cases} \quad (7)$$

In the above equations, $Da=K_1/L'^2$ is the Darcy number, $Ra=K_1g\beta L'^2q'/(k\alpha v)$ the Darcy-Rayleigh number based on permeability $K_1, K^*=K_1/K_2$ the permeability ratio and $\lambda=\mu_{eff}/\mu$ the relative viscosity.
In the present study $\lambda=1$ as a first approximation in Brinkman's extension for which $\mu_{eff}\approx\mu$. The quasi-state of the resulting transient natural convection heat transfer in the present study will be reached if the heating process is maintained long enough. So, all quantities governing the phenomenon become nearly independent of time ,except the temperature which continues to increase with. Consequently, assuming that $T=\tilde{T}-\xi t$, governing equations become at quasi-steady-state

$$a\frac{\partial^2\psi}{\partial x^2}+c\frac{\partial^2\psi}{\partial x\partial y}+b\frac{\partial^2\psi}{\partial y^2}=\lambda Da\nabla^4\psi-Ra\frac{\partial T}{\partial x} \quad (8)$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\partial T}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial\psi}{\partial x}+\xi \quad (9)$$

where ξ is a heat sink term.

We will consider the following boundary conditions:

- both horizontal boundaries rigid

$$\psi=\frac{\partial\psi}{\partial y}=0, \frac{\partial T}{\partial y}=-1 \text{ at } y=0 \quad (10)$$

$$\psi=\frac{\partial\psi}{\partial y}=0, \frac{\partial T}{\partial y}=0 \text{ at } y=1 \quad (11)$$

- both horizontal boundaries stress-free

$$\psi=\frac{\partial^2\psi}{\partial y^2}=0, \frac{\partial T}{\partial y}=-1 \text{ at } y=0 \quad (12)$$

$$\psi=\frac{\partial^2\psi}{\partial y^2}=0, \frac{\partial T}{\partial y}=-1 \text{ at } y=1 \quad (13)$$

3 Analytical solution

The appropriate solutions to the above equations can be sought in the form as in [6] and [3] :

$0\le x\le 1, \quad 0\le y\le 1-\delta$

$$\begin{cases} \psi(x,y)= & \psi_0(y) \\ T(x,y)= & Cx+\phi(y) \end{cases} \quad (14)$$

$0\le x\le 1, \ 1-\delta<y\le 1$

$$\begin{cases} \psi(x,y)=\psi_1(x+y) \\ T(x,y)=\omega(x+y)-\frac{y}{Ra}\varepsilon(1-y) \end{cases} \quad (15)$$

Where C is the unknown but constant temperature gradient in the x- direction, δ is the thickness of the boundary layer at $y=1$ [12] and

$$\varepsilon(z)=\begin{cases} 1 & z>0 \\ 0 & z=0 \\ -1 & z<0 \end{cases} \quad (16)$$

Substituting the functions (14) and (15) in the equations (8) and (9) , we obtain for $y<1$

$$\begin{cases} \psi_0'''' - b\tau^2\psi_0'' = \tau^2 RaC. \\ \phi'' = C\psi_0' + \xi \\ \psi_1'''' - \rho\psi_1'' = \frac{\tau^2}{4} Ra\omega'. \\ 2\omega'' = \frac{1}{Ra}\psi_1' + \xi \end{cases} \quad (17)$$

Where $\tau^2 = \frac{1}{Da}, \rho = \frac{\tau^2}{4}(a + c + b)$ and $'$ denotes the derivative. Without loss of generality we will consider $\xi = 0$ in the following discussion.

The solutions to the above equations are in the form

$$\psi_0(y) = RaCf(y) \quad (18)$$

$$\phi(y) = RaC^2 p(y) + q(y) \quad (19)$$

$$\psi_1(x + y) = RaCg(x + y) \quad (20)$$

$$\omega(x + y) = Ch(x + y) \quad (21)$$

Where f, g, h and p are functions depending upon the hydrodynamic and thermal boundary conditions imposed on the porous layer in the y direction and q is the temperature profile for pure conduction regime.

The solutions to equations (17) satisfying the boundary conditions (10),(11) can be defined by

$0 \le x \le 1, \ \ 0 \le y \le 1 - \delta$

$$f(y) = \frac{1}{2\tau b^{3/2}} \left\{ \frac{\cosh\left[\tau\sqrt{b}\left(y - \frac{1}{2}\right)\right]}{\sinh\frac{\tau\sqrt{b}}{2}} \right.$$

$$\left. - \coth\frac{\tau\sqrt{b}}{2} - \tau\sqrt{b}(y^2 - y) \right\}, \quad (22)$$

$$p(y) = \frac{1}{2\tau b^{3/2}} \left\{ \frac{\sinh\left[\tau\sqrt{b}\left(y - \frac{1}{2}\right)\right]}{\tau\sqrt{b}\sinh\frac{\tau\sqrt{b}}{2}} \right.$$

$$-y\coth\frac{\tau\sqrt{b}}{2} - \tau\sqrt{b}\left(\frac{y^3}{3} - \frac{y^2}{2}\right)$$

$$\left. \frac{-\tau\sqrt{b}}{3} - \frac{3}{\tau\sqrt{b}} + 2\coth\frac{\tau\sqrt{b}}{2} \right\}, \quad (23)$$

$$q(y) = -y \quad ; \quad (24)$$

$0 \le x \le 1, \ \ 1 - \delta < y \le 1$

$$g(x + y) = (\cos r(x + y) + \sin r(x + y)$$

$$+ \frac{1}{cosh2s}\cosh s\,(x + y)\Big\}\,\varepsilon(1 - y) \quad (25)$$

$$h(x + y) = \frac{1}{2}\left(\frac{1}{r}\sin r\,(x + y) - \frac{1}{r}\cos r(x + y)\right.$$

$$\left. + \frac{1}{scosh2s}\sinh s\,(x + y)\right)\varepsilon(1 - y) \quad (26)$$

Where

$$r = \left[\frac{1}{2}\left(\sqrt{\rho^2 + \frac{1}{2}\tau^2} - \rho\right)\right]^{\frac{1}{2}} \quad (27)$$

$$s = \left[\frac{1}{2}\left(\sqrt{\rho^2 + \frac{1}{2}\tau^2} + \rho\right)\right]^{\frac{1}{2}} \quad (28)$$

Then the stream function and temperature fields are known from the equations (14),(15) , (22) , (23), (24), (25) and (26).

Let us define the stream functions gradient $\frac{\widehat{\partial\psi}}{\partial y}, \frac{\overline{\partial\psi}}{\partial y}$ in the y direction and the function θ depending only on y as follows

$$\frac{\widehat{\partial\psi}}{\partial y}(y) = \int_0^1 \frac{\partial\psi}{\partial y}(x, y)dx, 0 \le y \le 1 - \delta \quad (29)$$

$$\frac{\overline{\partial\psi}}{\partial y}(y) = \int_0^1 \frac{\partial\psi}{\partial y}(x, y)dx, 1 - \delta < y \le 1 \quad (30)$$

$$\theta(y) = \int_0^1 [T(x, y) - Cx]dx, 1 - \delta < y \le 1. \quad (31)$$

Therefore

$$\frac{\widehat{\partial\psi}}{\partial y}(y) = RaCf'(y), \ \ 0 \le y \le 1 - \delta \quad (32)$$

$$\frac{\overline{\partial\psi}}{\partial y}(y) = RaC\eta(y), 1 - \delta < y \le 1 \quad (33)$$

$$\theta(y) = C\chi(y) - \frac{y}{Ra}\varepsilon(1 - y)\,, 1 - \delta < y \le 1 \quad (34)$$

Where

$$\eta(y) = [\cos r(1 + y) - \cos(ry)$$

$$+ \sin r(1 + y) - \sin(ry)$$

$$+ \frac{1}{cosh2s}\left(coshs(1 + y) - cosh(sy)\right)\Big]\,\varepsilon(1 - y)$$

$$\chi(y) = \frac{1}{2}\left\{-1 + \left[\frac{1}{r^2}(\cos(ry) - \cos r(1 + y))\right.\right.$$

$$+\frac{1}{r^2}(\sin(ry)-\sin r(1+y))$$

$$+\frac{1}{s^2\cosh 2s}\left(\cosh s(1+y)-\cosh sy\right)\Big]\varepsilon(1-y)\Big\}$$

Remark. If $T(x,y)=Cx+\phi(y)$, then

$$\theta(y)=\int_0^1[T(x,y)-Cx]dx=\int_0^1\phi(y)dx=\phi(y)$$

Following Bejan [2], an equivalent energy flux condition in the x direction can be imposed such that

$$C=\int_0^{1-\delta}\left(\frac{\partial\widetilde{\psi}}{\partial y}(y)\right)_x\phi(y)dy$$

$$+\int_{1-\delta}^{1}\left(\frac{\partial\overline{\psi}}{\partial y}(y)\right)_x\phi(y)dy. \qquad (35)$$

Substituting the equations (22), (23), (24), (25) and (26) into (35) we can write the following equation

$$C=Ra^2C^3I_2+RaC^2I_3+(RaI_1-I_4)C \qquad (36)$$

Where

$$I_1=\int_0^{1-\delta}f'(y)q(y)dy \qquad (37)$$

$$I_2=\int_0^{1-\delta}f'(y)p(y)dy \qquad (38)$$

$$I_3=\int_{1-\delta}^{1}\chi(y)\eta(y)dy \qquad (39)$$

$$I_4=\int_{1-\delta}^{1}y\eta(y)dy. \qquad (40)$$

Consider the discriminant

$$\Delta=Ra^2[I_3^2-4I_2(RaI_1-I_4-1)] \qquad (41)$$

of the equation (36), where $I_1>0, I_4>0, I_2<0$

When $\Delta>0$ (it is sufficient that $RaI_1>1+I_4$) the equation (36) admits the real roots C_1, C_2, C_3 :

$C_1=0$,

$$C_2=\frac{-RaI_3+\sqrt{\Delta}}{2Ra^2I_2},$$

$$C_2=\frac{-RaI_3-\sqrt{\Delta}}{2Ra^2I_2}.$$

It gives rise to convection cells in opposite directions.

When $\Delta<0$ or $I_3^2<4I_2(RaI_1-1-I_4)$ and $RaI_1<1+I_4$, the equation (36) admits the real root $C_1=0$ and there is no convection.

Obviously,the marginal state determining the critical Rayleigh number Ra_c for the onset of convection is reached when

$$Ra_cI_1=1+I_4\Leftrightarrow Ra_c=\frac{1}{I_1}+\frac{I_4}{I_1} \qquad (42)$$

The Nusselt number Nu is given by

$$Nu=\frac{\Delta T_c}{\Delta T} \qquad (43)$$

Where $\Delta T=T(0,0)-T(0,1)$ is the wall-to-wall dimensionless temperature difference and $\Delta T_c=q(0)-q(1)=1$ the corresponding value of pure conduction regime.

Taking into account the equations (32),(33) and (34) we can express I_1, I_2, I_3, I_4 as follows

$$I_1=\frac{1}{12b}\Bigg\{(1-\delta)^2(1-4\delta)+\frac{6}{\tau^2b}$$

$$-\frac{6}{\tau\sqrt{b}}\frac{(1-\delta)cosh\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{sinh\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{6}{\tau^2b}\frac{sinh\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{sinh\frac{\tau\sqrt{b}}{2}}\Bigg\}; \qquad (44)$$

$$I_2=\frac{1}{4\tau^2b^3}\Bigg\{\frac{1}{4\tau\sqrt{b}sinh^2\frac{\tau\sqrt{b}}{2}}$$

$$\times\left[sinh2\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)+sinh\tau\sqrt{b}\right.$$

$$\left.+2\tau\sqrt{b}(\delta-1)\right]+\Bigg[(\delta-1)coth\frac{\tau\sqrt{b}}{2}$$

$$+\frac{2(2\delta-1)}{\tau\sqrt{b}}+\frac{\tau\sqrt{b}}{6}(1-\delta)^2(1+2\delta)$$

$$-\frac{\tau\sqrt{b}}{3}-\frac{3}{\tau\sqrt{b}}+2coth\frac{\tau\sqrt{b}}{2}\Bigg]$$

$$\times \frac{cosh\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)}{sinh\frac{\tau\sqrt{b}}{2}}$$

$$+\left[\frac{coth\frac{\tau\sqrt{b}}{2}}{\tau\sqrt{b}}+\frac{4}{\tau^2 b}-\delta(1-\delta)\right]$$

$$\times \frac{sinh\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)}{sinh\frac{\tau\sqrt{b}}{2}}$$

$$+\left[\frac{\tau\sqrt{b}}{3}+\frac{3}{\tau\sqrt{b}}-2coth\frac{\tau\sqrt{b}}{2}-\frac{1}{\tau\sqrt{b}}\right.$$

$$\left.+\frac{\tau\sqrt{b}}{6}(1-\delta)^2(1-4\delta)\right]coth\frac{\tau\sqrt{b}}{2}$$

$$+\frac{4}{\tau^2 b}+\frac{\tau^2 b}{30}(1-\delta)^3(4\delta^2+2\delta-1)$$

$$+\tau\sqrt{b}\delta(1-\delta)\left(-\frac{\tau\sqrt{b}}{3}-\frac{3}{\tau\sqrt{b}}\right.$$

$$\left.+2coth\frac{\tau\sqrt{b}}{2}\right)\Big\}; \quad (45)$$

$$I_3=\frac{1}{2r}(\sin r(2-\delta)-\sin 2r+\sin r$$

$$-\sin r(1-\delta)+\cos 2r-\cos r(2-\delta)$$

$$+\cos r(1-\delta)-\cos r)$$

$$+\frac{1}{2s\ cosh 2s}(\sinh s(2-\delta)$$

$$-\sinh 2s+\text{sinhs}-\sinh s\,(1-\delta))$$

$$+\frac{\delta}{r^2}(\cos r-1)-\frac{1}{2r^3}((\sin 2r-\sin r)^2$$

$$-(\sin r(2-\delta)-\sin r\,(1-\delta))^2)$$

$$+\frac{2\sin h^2\frac{s}{2}}{s^2\cos h^2(2s)}$$

$$\times\left(-\delta+\frac{1}{2s}(\sinh 3s-\sinh s(3-2\delta))\right)$$

$$+\frac{2\sin\frac{r}{2}\sin h\frac{s}{2}}{s^2(r^2+s^2)\cos h2s}$$

$$\times\left[s\left(cos\frac{3r}{2}-\sin\frac{3r}{2}\right)\cos h\frac{3s}{2}\right.$$

$$+s\left(-\cos r\left(\frac{3}{2}-\delta\right)+\sin r\left(\frac{3}{2}-\delta\right)\right)\cosh s\left(\frac{3}{2}-\delta\right)$$

$$+r\left(\sin\frac{3r}{2}+\cos\frac{3r}{2}\right)\sin h\frac{3s}{2}+r\left(-\sin r\left(\frac{3}{2}-\delta\right)\right.$$

$$\left.\left.-\cos r\left(\frac{3}{2}-\delta\right)\right)\sin hs\left(\frac{3}{2}-\delta\right)\right]$$

$$+\frac{2\sin\frac{r}{2}\sinh\frac{s}{2}}{r^2(r^2+s^2)\cos h2s}\left[s\left(-\cos\frac{3r}{2}+\sin\frac{3r}{2}\right)\cosh\frac{3s}{2}\right.$$

$$+s\left(\cos r\left(\frac{3}{2}-\delta\right)-\sin r\left(\frac{3}{2}-\delta\right)\right)\cos hs\left(\frac{3}{2}-\delta\right)$$

$$-r\left(\sin\frac{3r}{2}+\cos\frac{3r}{2}\right)\sinh\frac{3s}{2}+r\left(\sin r\left(\frac{3}{2}-\delta\right)\right.$$

$$\left.\left.+\cos r\left(\frac{3}{2}-\delta\right)\right)\sin hs\left(\frac{3}{2}-\delta\right)\right]; \quad (46)$$

$$I_4=\frac{1}{r}[\sin 2r-\sin r+(1-\delta)(-\sin r(2-\delta)$$

$$+\sin r(1-\delta))]+\frac{1}{r}[-\cos 2r+\cos r$$

$$+(1-\delta)(\cos r(2-\delta)-\cos r(1-\delta))]$$

$$+\frac{1}{s\cos h2s}[\sin h2s-\sin hs+(1-\delta)\,(-\sin hs(2-\delta)$$

$$+\sin hs(1-\delta))]+\frac{1}{r^2}[\cos 2r-\cos r-\cos r(2-\delta)$$

$$+\cos r(1-\delta))]+\frac{1}{r^2}[\sin 2r-\sin r-\sin r(2-\delta)$$

$$+\sin r\,(1-\delta)]+\frac{1}{s^2\cos h2s}[-\cos h2s+\cos h\,2s$$

$$+\cos hs(2-\delta)-\cos hs(1-\delta)] \quad (47)$$

Therefore the temperature gradient C may be evaluated from (42) for given values of Ra, Da, K^* and θ

We can express the critical Rayleigh number Ra_c in the form

$$Ra_c = 12b\Bigg[(1-\delta)^2(1-4\delta)+\frac{6}{r^2 b}$$

$$-\frac{6}{\tau\sqrt{b}}\frac{(1-\delta)\cos h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\sinh\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{6}{\tau^2 b}\frac{\sin h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\sinh\frac{\tau\sqrt{b}}{2}}\Bigg]^{-1}$$

$$+12b\left\{\frac{1}{r}[\sin 2r-\sin r\right.$$

$$+(1-\delta)(-\sin r(2-\delta)+\sin r(1-\delta))]$$

$$+\frac{1}{r}[-\cos 2r+\cos r$$

$$+(1-\delta)(\cos r(2-\delta)-\cos r(1-\delta)]$$

$$+\frac{1}{s\cos h2s}[\sin h2s-\sin hs$$

$$+(1-\delta)(-\sin hs(2-\delta)+\sin hs(1-\delta))]$$

$$+\frac{1}{r^2}[+\cos 2r-\cos r-\cos r(2-\delta)$$

$$+\cos r(1-\delta)]+\frac{1}{r^2}[\sin 2r-\sin r$$

$$-\sin r(2-\delta)+\sin r(1-\delta)]$$

$$+\frac{1}{s^2\cos h2s}[-\cos h2s+\cos hs$$

$$\left.+\cos hs(2-\delta)-\cos hs(1-\delta)\right\}$$

$$\times\Bigg\{(1-\delta)^2(1-4\delta)+\frac{6}{r^2 b}$$

$$-\frac{6}{\tau\sqrt{b}}\frac{(1-\delta)\cos h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\sinh\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{6}{\tau^2 b}\frac{\sin h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\sin h\frac{\tau\sqrt{b}}{2}}\Bigg]^{-1} \quad (48)$$

Remark . When $\psi_1(x+y)=0$, the quantities I_3, I_4 in (39) , (40) vanish and for $\delta=0$, the quantity I_2 and the critical Rayleigh number $Rac=\frac{1}{I_1}$ are obtained by Degan and Vasseur [6]

Now we can point out two limiting cases:

When τ approaches ∞ (corresponding to the anisotropic porous medium),

$$Ra_c=\frac{12b}{(1-\delta)^2(1-4\delta)}$$

When τ approaches 0 (i.e.viscous fluid case where anisotropic effects are irrelevant),

$$Ra_c^*=1440.$$

For

$$\delta=\frac{3}{4}+\left(\frac{-5}{64}+\frac{1}{16}\sqrt{\frac{3}{2}}\right)^{\frac{1}{3}}-\left(\frac{5}{64}+\frac{1}{16}\sqrt{\frac{3}{2}}\right)^{\frac{1}{3}}$$

$$Ra_c=24b$$

This result is obtained by [6].

The solutions to the equations (17) satisfying the boundary conditions (12) and (13) are given by

$$0\le x\le 1\ ,0\le y\le 1-\delta$$

$$f(y)=\frac{1}{(\tau b)^2}\left\{\frac{\cos h\left[\tau\sqrt{b}\left(y-\frac{1}{2}\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}}\right.$$

$$\left.+\frac{b\tau^2}{2}(y-y^2)-1\right];\quad (49)$$

$$p(y)=\frac{1}{b^2\tau^2}\left\{\frac{\sin h\left[\tau\sqrt{b}\left(y-\frac{1}{2}\right)\right]}{\tau\sqrt{b}\cos h\frac{\tau\sqrt{b}}{2}}\right.$$

$$\left.-y+\frac{\tau^2 b}{2}\left(\frac{y^2}{2}-\frac{y^3}{3}\right)+B\right\},\quad (50)$$

$$q(y)=-y\,,\quad (51)$$

$$0\le x\le 1,\ \ 1-\delta<y\le 1$$

$$g(x+y)=\{\cos r(x+y)+\sin r(x+y)$$

$$\left.+\frac{\sin h[s(x+y)]}{\cos h2s}\right\}\varepsilon(1-y)\quad (52)$$

$$h(x+y)=\frac{1}{2}\left\{\frac{1}{r}\sin[r(x+y)]-\frac{1}{r}\cos[r(x+y)]\right.$$

$$\left.+\frac{\cos hs(x+y)}{s\cos h2s}\right\}\varepsilon(1-y)\quad (53)$$

Where B will be defined with respect to Nusselt number.

Taking into account the equations (49), we can express the equations (37) and (40) as follows

$$I_1 = \frac{1}{12b}\{(1-\delta)^2(1-4\delta)$$

$$+\frac{12(\delta-1)}{\tau^2 b}\frac{\cos h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{12}{\tau^3 b^{\frac{3}{2}}}\left(\frac{\sin h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}}\right.$$

$$\left.+\tan h\frac{\tau\sqrt{b}}{2}\right)\Big\} \; ; \qquad (54)$$

$$I_4 = \frac{1}{r}[\sin(2r) - \sin r + (\delta-1)(\sin r(2-\delta)$$

$$-\sin r(1-\delta))] + \frac{1}{r}[-\cos(2r) + \cos r$$

$$+(\delta-1)\big(-\cos\big(r(2-\delta)\big) + \cos r(1-\delta)\big)\big]$$

$$+\frac{1}{s\cos h2s}[\cos h(2s) - \cos h\, s$$

$$+(\delta-1)(\cos(s(2-\delta)) - \cosh s(1-\delta)\big)\big]$$

$$+\frac{1}{r^2}[\cos(2r) - \cos r - \cos h\, r(2-\delta)$$

$$+\cos r(1-\delta)] + \frac{1}{r^2}[\sin(2r)$$

$$-\sin r - \sin\big(r(2-\delta)\big) + \sin r(1-\delta)\big]$$

$$+\frac{1}{s^2\cos h2s}[-\sin h2s + \sin hs$$

$$+\sin h\big(s(2-\delta)\big) - \sin hs(1-\delta)\big] \qquad (55)$$

Therefore the temperature gradient C may be evaluated from (42) for given values of Ra, Da, K^* and θ.

We can express the critical Rayleigh number Ra_c in the form

$$Ra_c = 12b[(1-\delta)^2(1-4\delta)$$

$$+\frac{12(\delta-1)}{\tau^2 b}\frac{\cos h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{12}{\tau^3 b^{\frac{3}{2}}}\left(\frac{\sin h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}} + \tan h\frac{\tau\sqrt{b}}{2}\right)\Bigg]^{-1}$$

$$+12b\Big[\frac{1}{r}(\sin(2r) - \sin r + (\delta-1)(\sin r(2-\delta)$$

$$-\sin r(1-\delta))\big) + \frac{1}{r}(-\cos(2r) + \cos r$$

$$+(\delta-1)(-\cos r(2-\delta)) + \cos r(1-\delta))\big)$$

$$+\frac{1}{s\cos h\, 2s}(\cos h(2s) - \cos hs$$

$$+(\delta-1)(\cos(s(2-\delta)) - \cos hs(1-\delta))\big)$$

$$+\frac{1}{r^2}(\cos(2r) - \cos r - \cos r(2-\delta)$$

$$+\cos r(1-\delta)) + \frac{1}{r^2}(\sin(2r)$$

$$-\sin r - \sin\big(r(2-\delta)\big) + \sin r(1-\delta)\big)$$

$$+\frac{1}{s^2\cos h\, 2s}(-\sinh 2s + \sin hs + \sin h(s(2-\delta))$$

$$-\sin hs(1-\delta))\big)\big][(1-\delta)^2(1-4\delta)$$

$$+\frac{12(\delta-1)}{\tau^2 b}\frac{\cos h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}}$$

$$+\frac{12}{\tau^3 b^{\frac{3}{2}}}\left(\frac{\sin h\left[\tau\sqrt{b}\left(\frac{1}{2}-\delta\right)\right]}{\cos h\frac{\tau\sqrt{b}}{2}} + \tan h\frac{\tau\sqrt{b}}{2}\right)\Bigg]^{-1} \qquad (56)$$

4 Result and discussion

In this paper we have investigated the anisotropic effect of the porous medium.

When the dimensionless Darcy parameter is infinitely large (corresponding to the anisotropic porous medium), the critical Darcy-Rayleigh is a function of the permeability ratio and the orientation angle . When the dimensionless Darcy parameter approaches 0 (corresponding to the viscous fluid where anisotropic effect is irrelevant) the critical Darcy-

Rayleigh is constant. Moreover for the appropriated value of the thickness the critical Darcy-Rayleigh is that obtained in [6]

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